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Noether symmetry of the single-particle system

Zhu Dongpei

Department of Physics, University of Wisconsin-Madison, Madison, Wisconsin 53706 USA
and
Department of Modern Physics, China University of Science and Technology, Hefei, Anhui, China

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Abstract. The Noether-type point transformation symmetry of the one-dimensional single-particle system is investigated systematically. All four possible potentials which possess symmetry larger than time translation have been found. The connection between symmetries on the classical level and the quantum level is also established.

1. Introduction

The symmetry group of a specific single-particle system has been studied by many authors. Lie (1888) first obtained the generators of the invariance group of a free particle. For a harmonic oscillator, there are several papers contributing to the analysis of a symmetry group on both the classical and the quantum mechanics levels (Anderson and Davison 1974, Wulfman and Wybourne 1976, Lutzky 1978a, b).

It is well known that all stationary conservative systems possess the symmetry of time translation, and that the corresponding conserved quantity is the energy of the system. In this paper we refer to this symmetry as a trivial one. Is there any other system beyond the free particle and the harmonic oscillator possessing non-trivial symmetry, and what symmetry is that? Our purpose is to answer this question.

In this paper we shall systematically investigate the symmetry group of a one-dimensional, stationary, conservative single-particle system. We shall find all potentials which possess a symmetry bigger than a mere time translation. We shall use Lutzky's formulation of Noether's theorem (Noether 1918, Gelfand and Fomin 1963, Lutzky 1978a, b) because it is very convenient for searching for all Noether-type symmetries and their corresponding conserved quantities in a system. Therefore it facilitates passage to quantum mechanics.

In § 2, for a general system (i.e. a particle moving in a time-independent potential) we derive a set of equations which the symmetry transformation and the potential should satisfy. We also write down the general form of the conserved quantities of such a system. In § 3, we prove a symmetry correspondence theorem which relates the symmetries on the classical level with those on the quantum level. Section 4 provides the most general solution to the problem: four kinds of potentials, the corresponding non-trivial Noether-type symmetry and the conserved quantities. We also derive the algebra in which the symmetry generators are closed for both the classical level and the quantum level. To this end, we give a detailed demonstration of the theorem proven in § 3. Finally, we conclude this paper in § 5 with a discussion of our results.

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2. Symmetry and the potential equation

The system we investigate has the following Lagrangian:

$$\mathscr{L}=\tfrac{1}{2}\dot{q}^2-V(q). \tag{1}$$

Here we have chosen a unit mass for the particle.

Suppose the action of this system is invariant under certain point transformations; we then obtain a symmetry transformation of the system. This symmetry can be described by a first extended one-parameter Lie group. The transformation can be written in the form

$$\begin{pmatrix} T \\ Q \\ \dot{Q} \end{pmatrix}=\exp(\theta E(q,\dot{q},t))\begin{pmatrix} t \\ q \\ \dot{q} \end{pmatrix} \tag{2}$$

where θ is the group parameter and E is the infinitesimal operator of the first extended group

$$E(q,\dot{q},t)=G(q,t)+(\dot{\eta}-\dot{q}\dot{\xi})\partial/\partial\dot{q} \qquad G(q,t)\equiv\xi(q,t)\partial/\partial t+\eta(q,t)\partial/\partial q \tag{3}$$

while G is the generator of the one-parameter group (Cohen 1931). The dot here denotes a complete time differentiation.

If the transformation (2) satisfies the following equation

$$E\{\mathscr{L}\}=-\dot{\xi}\mathscr{L}+\dot{f} \tag{4}$$

we obtain a Noether-conserved quantity

$$\begin{aligned}\Phi&=(\xi\dot{q}-\eta)\partial\mathscr{L}/\partial\dot{q}-\xi\mathscr{L}+f(q,t)\\&=\xi H-\tfrac{1}{2}(\eta p+p\eta)+f.\end{aligned} \tag{5}$$

Here f is a function of time and coordinate. H is the Hamiltonian of the system

$$H=\dot{q}\partial\mathscr{L}/\partial\dot{q}-\mathscr{L}=\tfrac{1}{2}p^2+V(q) \qquad p=\dot{q}. \tag{6}$$

For convenience in passing to quantum mechanics, we have already symmetrised the expression for the conserved quantity.

From equations (1), (3) and (4) we have the following set of equations:

$$\xi_q=0 \qquad \xi_t-2\eta_q=0 \qquad -V\xi_q+\eta_t-f_q=0 \qquad V\xi_t+\eta V_q+f_t=0. \tag{7}$$

In matrix form, we can rewrite equation (7) as follows

$$S\psi=0 \tag{8}$$

$$\psi=\begin{pmatrix} \xi(t) \\ \eta(q,t) \\ f(q,t) \end{pmatrix} \tag{9}$$

$$S=\begin{pmatrix} \partial/\partial t & -2\partial/\partial q & 0 \\ 0 & \partial/\partial t & -\partial/\partial q \\ V\partial/\partial t & V_q & \partial/\partial t \end{pmatrix}. \tag{10}$$

Here ξ is a function of time only.

Equation (7) can be used for two purposes. Firstly, for a given potential, we can use it to find out the corresponding symmetry transformations. Then equation (7) is a set of homogeneous linear equations of (ξ, η, f). We can apply the superposition principle to obtain the total solution, namely, the general solution ψ is a linear combination of all linear independent solutions ψ_i:

$$\psi = \sum_i A_i \psi_i \tag{11}$$

$$\psi_i = \begin{pmatrix} \xi_i \\ \eta_i \\ f_i \end{pmatrix}. \tag{12}$$

For every solution ψ_i, there is a corresponding generator G_i which generates the symmetry group on the classical level

$$G_i = \xi_i \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial q} \tag{13}$$

and a corresponding constant of motion

$$\Phi_i = \xi_i H - \tfrac{1}{2}(\eta_i p + p\eta_i) + f_i \tag{14}$$

which generates the symmetry group on the quantum level.

Conversely, for a given symmetry transformation, we can use equation (7) to determine the potentials which possess such symmetry. Fortunately, equation (7) can help us to determine both the symmetry transformation and the potential simultaneously, so that we can solve the symmetry problem of one-dimensional single-particle systems completely.

3. The symmetry correspondence theorem

Before we solve equation (7) or (8), we prove a theorem which relates the symmetry between classical mechanics and quantum mechanics.

Firstly, one may check that if two sets of (ξ, η, f)

$$\psi_i = \begin{pmatrix} \xi_i \\ \eta_i \\ f_i \end{pmatrix} \qquad \psi_j = \begin{pmatrix} \xi_j \\ \eta_j \\ f_j \end{pmatrix}$$

are the solutions of equation (8)

$$S\psi_i = 0 \qquad S\psi_j = 0. \tag{15}$$

Then the combination

$$\psi_{ij} = G_i \psi_j - G_j \psi_i \tag{16}$$

is also a solution of equation (8)

$$S\psi_{ij} = 0. \tag{17}$$

ψ_{ij} can be expressed as a linear combination of all linearly independent solutions of equation (18)

$$G_i\psi_j - G_j\psi_i = \sum_k C_{ij}^k \psi_k. \tag{18}$$

More explicitly,

$$\begin{aligned}
&\xi_i\xi_{jt} - \xi_j\xi_{it} + \eta_i\xi_{jq} - \eta_j\xi_{iq} = \sum_k C_{ij}^k \xi_k \\
&\xi_i\eta_{jt} - \xi_j\eta_{it} + \eta_i\eta_{jq} - \eta_j\eta_{iq} = \sum_k C_{ij}^k \eta_k \\
&\xi_i f_{jt} - \xi_j f_{it} + \eta_i f_{jq} - \eta_j f_{iq} = \sum_k C_{ij}^k (f_k + d_k).
\end{aligned} \tag{19}$$

Here d_k is a constant. The reason for its appearance is that we can only determine f_k up to a constant by equation (7) and it does not have any classical effect. Actually, we can determine ξ accurately only up to a constant as well, which reflects the fact that any stationary system possesses the time translation symmetry, but it can be pinned down by the classical symmetry requirement.

The first two equations of (19) are indeed the integrable condition of the Lie group

$$[G_i, G_j] = \sum_k C_{ij}^k G_k. \tag{20}$$

From this, the integrable condition of the first extended Lie group can be easily derived

$$[E_i, E_j] = \sum_k C_{ij}^k E_k. \tag{21}$$

Passing to quantum mechanics, we have to use the canonical quantisation condition ($\hbar = 1$)

$$[q, p] = \mathrm{i}. \tag{22}$$

Direct calculation yields, using equation (19),

$$[\Phi_i, \Phi_j] = \mathrm{i} \sum_k C_{ij}^k (\Phi_k + d_k I) \tag{23}$$

where I is the identity operator.

Comparing equations (23) and (20), we may formulate the following symmetry correspondence theorem. 'Up to the identity operator, a quantum system possesses the same symmetry as the Noether-type symmetry of the same classical system.'

The reason for the appearance of the identity operator in (23) is obvious: instead of an Abelian algebra

$$[\partial/\partial t, \partial/\partial q] = 0 \tag{24}$$

in the classical case, we have the Heisenberg algebra (22) of quantum mechanics.

Finally, a similar calculation shows, using equation (7), that the classical conserved quantities Φ_i are also conserved in quantum mechanics

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_i = \frac{\partial \Phi_i}{\partial t} + \mathrm{i}[H, \Phi_i] = 0. \tag{25}$$

4. Potentials and symmetry algebras

Now we turn to solving equation (7) to find out all possible potentials and their corresponding symmetries.

Equation (7) can first be integrated

$$\xi = \xi(t) \qquad \eta = \tfrac{1}{2}\dot{\xi}q + \eta_0(t) \qquad f = \tfrac{1}{4}\ddot{\xi}q^2 + \dot{\eta}_0 q + f_0(t). \tag{26}$$

Then the pure time functions ξ, η_0 and f_0 satisfy the following equation

$$V\dot{\xi} + \tfrac{1}{2}qV_q\dot{\xi} + \eta_0 V_q = -\tfrac{1}{4}q^2\dddot{\xi} - q\ddot{\eta}_0 - \dot{f}_0. \tag{27}$$

In the trivial case $\dot{\xi} = 0$ and $\eta_0 = 0$ (i.e. only time translation), V can be any potential. For $\dot{\xi} \neq 0$, this equation constrains the form of V. Regarding equation (27) as an equation for V, we have a linear, first-order, inhomogeneous differential equation. The most general solution is of the form

$$V(q) = a_{-2}/(q+\lambda)^2 + a_2(q+\lambda)^2 + a_1(q+\lambda) + a_0 \tag{28}$$

where λ is independent of the coordinate q. The first term is the solution to the homogeneous equation, and a_{-2} will remain a free parameter. The other terms are a particular integral; the coefficients a_0, a_1 and a_2 can be fixed by putting (28) back into equation (27), which yields a set of equations for transformation functions:

$$\begin{aligned} &a_{-2}(\lambda\dot{\xi} - 2\eta_0) = 0 \qquad \dddot{\xi} + 8a_2\dot{\xi} = 0 \\ &\ddot{\eta}_0 + 2a_2\eta_0 - \tfrac{1}{2}\dddot{\xi} + (\tfrac{3}{2}a_1 - \lambda a_2)\dot{\xi} = 0 \\ &\dot{f}_0 + a_1\eta_0 - \lambda\ddot{\eta}_0 + \tfrac{1}{4}\lambda^2\dddot{\xi} + (a_0 - \tfrac{1}{2}\lambda a_1)\dot{\xi} = 0. \end{aligned} \tag{29}$$

This set of linear differential equations provides four kinds of solutions.

4.1. *Linear potential*

$$a_{-2} = 0 \qquad a_2 = 0. \tag{30}$$

The potential is

$$V = a_1(q+\lambda) + a_0. \tag{31}$$

The symmetry transformation functions are

$$\begin{aligned} &\xi = A_2t^2 + A_1t + A_0 \\ &\eta = A_2(qt - \tfrac{1}{2}a_1t^3) + A_1(\tfrac{1}{2}q - \tfrac{3}{4}a_1t^2) + A_5t + A_6 \\ &f = A_2F_1 + A_1F_1' + A_5(q - \tfrac{1}{2}a_1t^2) + A_6(-a_1t) + d \\ &F_1 = \tfrac{1}{2}q^2 - \tfrac{3}{2}a_1t^2q - (a_0 + \lambda a_1)t^2 + \tfrac{1}{8}a_1^2t^4 \\ &F_1' = -\tfrac{3}{2}a_1qt - (a_0 + \lambda a_1)t + \tfrac{1}{4}a_1^2t^3. \end{aligned} \tag{32}$$

Here A are arbitrary constants (group parameters).

We obtain five generators of the Lie group

$$G_1 = t^2 \frac{\partial}{\partial t} + (qt - \tfrac{1}{2}a_1 t^3)\frac{\partial}{\partial q}$$
$$G_2 = t\frac{\partial}{\partial t} + (\tfrac{1}{2}q - \tfrac{3}{4}a_1 t^2)\frac{\partial}{\partial q} \tag{33}$$
$$G_3 = \frac{\partial}{\partial t} \qquad G_4 = t\frac{\partial}{\partial q} \qquad G_5 = \frac{\partial}{\partial q}$$

which give the following commutative relations

$$\begin{array}{llllll} [G_i, G_j] & G_1 & G_2 & G_3 & G_4 & G_5 \\ G_1 & 0 & -G_1 & -2G_2 & 0 & -G_4 \\ G_2 & & 0 & -G_5 - \tfrac{3}{2}a_1 G_4 & \tfrac{1}{2}G_4 & -\tfrac{1}{2}G_5 \\ G_3 & & & 0 & G_5 & 0 \\ G_4 & & & & 0 & 0 \\ G_5 & & & & & 0 \end{array} \tag{34}$$

Corresponding to the symmetry transformations, we have the following constants of the motion

$$\Phi_1 = t^2 H_1 - Dt + \tfrac{1}{2}a_1 t^3 p + F_1$$
$$\Phi_2 = tH_1 - \tfrac{1}{2}D + \tfrac{3}{4}a_1 t^2 p + F_1' \tag{35}$$
$$\Phi_3 = H_1 - a_0 - \lambda a_1 \qquad \Phi_4 = -pt + q - \tfrac{1}{2}a_1 t^2 \qquad \Phi_5 = -p - a_1 t.$$

Here the Hamiltonian and dilatation operator are

$$H_1 = \tfrac{1}{2}p^2 + a_1(q + \lambda) + a_0 \qquad D = \tfrac{1}{2}(qp + pq). \tag{36}$$

Among these conserved quantities only two are independent. In fact, we have the relations

$$\Phi_1 = \tfrac{1}{2}\Phi_4^2 \qquad \Phi_2 = \tfrac{1}{4}(\Phi_4\Phi_5 + \Phi_5\Phi_4) \qquad \Phi_3 = \tfrac{1}{2}\Phi_5^2 + a_1\Phi_4. \tag{37}$$

In quantum mechanics the conserved quantities (35) satisfy the following algebra

$$\begin{array}{llllll} & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \\ \Phi_1 & 0 & -\mathrm{i}\Phi_1 & -2\mathrm{i}\Phi_2 & 0 & -\mathrm{i}\Phi_4 \\ \Phi_2 & & 0 & -\mathrm{i}(\Phi_3 - \tfrac{3}{2}a_1\Phi_4) & \mathrm{i}\tfrac{1}{2}\Phi_4 & -\mathrm{i}\tfrac{1}{2}\Phi_5 \\ \Phi_3 & & & 0 & \mathrm{i}\Phi_5 & -\mathrm{i}a_1 \\ \Phi_4 & & & & 0 & -\mathrm{i} \\ \Phi_5 & & & & & 0 \end{array} \tag{38}$$

This algebra contains an SO(2, 1) as a subalgebra. To show this we choose another set of generators J_i (α is an arbitrary real number)

$$J_1 = \alpha\Phi_1 - \tfrac{1}{4}\alpha^{-1}(\Phi_3 + \Phi_4) \qquad J_2 = \Phi_2 \qquad J_3 = \alpha\Phi_1 + \tfrac{1}{4}\alpha^{-1}(\Phi_3 + \Phi_4) \tag{39}$$

which satisfy the following SO(2, 1) algebra

$$[J_1, J_2] = -\mathrm{i}J_3 \qquad [J_2, J_3] = \mathrm{i}J_1 \qquad [J_3, J_1] = \mathrm{i}J_2. \tag{40}$$

Obviously the Hamiltonian H_1 of the system is not the compact generator of this subgroup.

4.2. Oscillator

$$a_{-2} = 0 \qquad a_2 = \tfrac{1}{2}\omega^2 \neq 0 \tag{41}$$

$$V = a_2(q+\lambda)^2 + a_1(q+\lambda) + a_0. \tag{42}$$

This case has been well studied by Lutzky (1978a). The symmetry functions are

$$\begin{aligned}
\xi &= A_3 \cos 2\omega t + A_4 \sin 2\omega t + A_0 \\
\eta &= -A_3[\omega q + \omega^{-1}(a_1 + \lambda\omega^2)] \sin 2\omega t + A_4[\omega q + \omega^{-1}(a_1 + \lambda\omega^2)] \cos 2\omega t \\
&\quad + A_7 \cos \omega t + A_8 \sin \omega t \\
f &= -F_2(A_3 \cos 2\omega t + A_4 \sin 2\omega t) - [\omega q + \omega^{-1}(a_1 + \lambda\omega^2)](A_7 \sin \omega t - A_8 \cos \omega t) + d \\
F_2 &= \omega^2 q^2 + 2q(a_1 + \lambda\omega^2) + \tfrac{1}{2}\lambda^2\omega^2 + \lambda a_1 + a_0 + \tfrac{1}{2}\omega^{-2}(a_1 + \lambda\omega^2)^2
\end{aligned} \tag{43}$$

which give us five generators

$$\begin{aligned}
G_1 &= \cos 2\omega t \partial/\partial t - [\omega q + \omega^{-1}(a_1 + \lambda\omega^2)] \sin 2\omega t \partial/\partial q \\
G_2 &= \sin 2\omega t \partial/\partial t + [\omega q + \omega^{-1}(a_1 + \lambda\omega^2)] \cos 2\omega t \partial/\partial q \\
G_3 &= \partial/\partial t \qquad G_4 = \cos \omega t \partial/\partial q \qquad G_5 = \sin \omega t \partial/\partial q.
\end{aligned} \tag{44}$$

These generators close on an algebra

$$\begin{array}{cccccc}
 & G_1 & G_2 & G_3 & G_4 & G_5 \\
G_1 & 0 & 2\omega G_3 & 2\omega G_2 & \omega G_5 & \omega G_4 \\
G_2 & & 0 & -2\omega G_1 & -\omega G_4 & \omega G_5 \\
G_3 & & & 0 & \omega G_5 & \omega G_4 \\
G_4 & & & & 0 & 0 \\
G_5 & & & & & 0.
\end{array} \tag{45}$$

The corresponding conserved quantities read

$$\begin{aligned}
\Phi_1 &= \cos 2\omega t(H_2 - F_2) + [\omega D + \omega^{-1}(a_1 + \lambda\omega^2)p] \sin 2\omega t \\
\Phi_2 &= \sin 2\omega t(H_2 - F_2) - [\omega D + \omega^{-1}(a_1 + \lambda\omega^2)p] \cos 2\omega t \\
\Phi_3 &= H_2 - a_0 + a_1^2/2\omega^2 \\
\Phi_4 &= -p \cos \omega t - [\omega q + \omega^{-1}(a_1 + \lambda\omega^2)] \sin \omega t \\
\Phi_5 &= -p \sin \omega t + [\omega q + \omega^{-1}(a_1 + \lambda\omega^2)] \cos \omega t.
\end{aligned} \tag{46}$$

They are interrelated by

$$\Phi_1 = \tfrac{1}{2}(\Phi_4^2 - \Phi_5^2) \qquad \Phi_2 = \tfrac{1}{2}(\Phi_4\Phi_5 + \Phi_5\Phi_4) \qquad \Phi_3 = \tfrac{1}{2}(\Phi_4^2 + \Phi_5^2). \tag{47}$$

These conserved quantities provide the following commutative relations on the quantum level

$$\begin{array}{c|ccccc} & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \\ \Phi_1 & 0 & 2i\omega\Phi_3 & 2i\omega\Phi_2 & i\omega\Phi_5 & i\omega\Phi_4 \\ \Phi_2 & & 0 & -2i\omega\Phi_1 & -i\omega\Phi_4 & i\omega\Phi_5 \\ \Phi_3 & & & 0 & -i\omega\Phi_5 & i\omega\Phi_4 \\ \Phi_4 & & & & 0 & i\omega \\ \Phi_5 & & & & & 0. \end{array} \tag{48}$$

We may see from (48) that the three generators Φ_1, Φ_2 and Φ_3 form an SO(2, 1) subalgebra with the Hamiltonian operator as a compact one.

4.3. *Centrifugal (centripetal) potential*

$$a_{-2} \neq 0 \qquad a_2 = 0. \tag{49}$$

Whenever a_{-2} is not vanishing, the a_1 should be zero. The potential has the form

$$V = a_{-2}/(q+\lambda)^2 + a_0. \tag{50}$$

The symmetry functions, the classical generators, their commutative relations, the conserved quantities, and their algebra are listed as follows:

$$\begin{aligned} \xi &= A_2 t^2 + A_1 t + A_0 \\ \eta &= A_2(q+\lambda)t + A_1 \tfrac{1}{2}(q+\lambda) \\ f &= A_2(\tfrac{1}{2}q^2 + \lambda q - a_0 t^2) - A_1 a_0 t + d \end{aligned} \tag{51}$$

$$\begin{aligned} G_1 &= t^2 \partial/\partial t + (q+\lambda)\partial/\partial q \\ G_2 &= t\partial/\partial t + \tfrac{1}{2}(q+\lambda)\partial/\partial q \\ G_3 &= \partial/\partial t \end{aligned} \tag{52}$$

$$\begin{array}{c|ccc} & G_1 & G_2 & G_3 \\ G_1 & 0 & -G_1 & -2G_2 \\ G_2 & & 0 & -G_3 \\ G_3 & & & 0 \end{array} \tag{53}$$

$$\begin{aligned} \Phi_1 &= t^2 H_3 - Dt - \lambda t p + \tfrac{1}{2}q^2 + \lambda q - a_0 t^2 + \tfrac{1}{2}\lambda^2 \\ \Phi_2 &= tH_3 - \tfrac{1}{2}D - \tfrac{1}{2}\lambda p - a_0 t \\ \Phi_3 &= H_3 - a_0 \\ H_3 &= \tfrac{1}{2}p^2 + a_{-2}/(q+\lambda)^2 + a_0 \end{aligned} \tag{54}$$

$$\begin{array}{c|ccc} & \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_1 & 0 & -i\Phi_1 & -2i\Phi_2 \\ \Phi_2 & & 0 & -i\Phi_3 \\ \Phi_3 & & & 0. \end{array} \tag{55}$$

We may observe an SO(2, 1) symmetry, but the Hamiltonian H_3 is not the compact operator.

4.4. Mixed potential

$$a_{-2} \neq 0 \qquad a_2 = \tfrac{1}{2}\omega^2 \neq 0 \tag{56}$$

$$V = a_{-2}/(q+\lambda)^2 + a_2(q+\lambda)^2 + a_0. \tag{57}$$

As in the above case, we list all the usual quantities and relations:

$$\begin{aligned}
&\xi = A_3 \cos 2\omega t + A_4 \sin 2\omega t + A_0 \\
&\eta = -A_3\omega(q+\lambda) \sin 2\omega t + A_4\omega(q+\lambda) \cos 2\omega t \\
&f = -A_3F_4 \cos 2\omega t - A_4F_4 \sin 2\omega t + d \\
&F_4 = \omega^2 q^2 + 2\lambda\omega^2 q + 2\lambda^2\omega^2 + a_0
\end{aligned} \tag{58}$$

$$\begin{aligned}
&G_1 = \cos 2\omega t\partial/\partial t - \omega(q+\lambda) \sin 2\omega t\partial/\partial q \\
&G_2 = \sin 2\omega t\partial/\partial t + \omega(q+\lambda) \cos 2\omega t\partial/\partial q \\
&G_3 = \partial/\partial t
\end{aligned} \tag{59}$$

$$\begin{array}{cccc}
 & G_1 & G_2 & G_3 \\
G_1 & 0 & 2\omega G_3 & 2\omega G_2 \\
G_2 & & 0 & -2\omega G_1 \\
G_3 & & & 0
\end{array} \tag{60}$$

$$\begin{aligned}
&\Phi_1 = \cos 2\omega t(H_4 - F_4) - \omega(D + \lambda p) \sin 2\omega t \\
&\Phi_2 = \sin 2\omega t(H_4 - F_4) + \omega(D + \lambda p) \cos 2\omega t \\
&\Phi_3 = H_4 - a_0 \\
&H_4 = \tfrac{1}{2}p^2 + a_{-2}/(q+\lambda)^2 + a_2(q+\lambda)^2 + a_0
\end{aligned} \tag{61}$$

$$\begin{array}{cccc}
 & \Phi_1 & \Phi_2 & \Phi_3 \\
\Phi_1 & 0 & -2\mathrm{i}\omega\Phi_3 & -2\mathrm{i}\omega\Phi_2 \\
\Phi_2 & & 0 & 2\mathrm{i}\omega\Phi_1 \\
\Phi_3 & & & 0.
\end{array} \tag{62}$$

The symmetry group is SO(2, 1) and the Hamiltonian H_4 stands as a compact generator.

In concluding this section, we should like to emphasise that these are all solutions of equation (7). One may use another method to check this: expanding (19) in a Laurent series about the origin of coordinates, one obtains an infinite set of equations. Solving these equations, one obtains the same answers.

5. Discussion

Section 4 tells us that all possible potentials which possess symmetry larger than time translation fall into four categories: linear (including free particles), oscillator, centrifugal potential and mixed potential. The ensuing symmetry of the classical system is not a complete symmetry of the equations of motion, because we are only dealing with the Noether-type point transformation symmetry. For example, the free particle possesses a symmetry generated by eight generators, but among them only five independent combinations lead to constants of motion. The same thing is true for the oscillator. For the centrifugal potential and the mixed potential we cannot apply the usual procedure to determine the full symmetry of the equations of motion, because of the singularity of the potential. Of course, on the quantum level the symmetry we obtain is a completely finite one.

The obvious symmetry of a stationary system is time translation. The minimal non-trivial symmetry, if it exists, is SO(2, 1). This can be seen from § 4. Group theory tells us that only the compact generator of SO(2, 1) can have a discrete spectrum. So in the linear and the centrifugal case the Hamiltonian of the system has a continuous spectrum only, while in the other two cases where the oscillator is involved the system has a discrete energy spectrum. On the classical level this means that in the later two cases the particle is subject to periodic motion.

In § 4 we have taken a_2 to be a positive number. If a_2 is negative the symmetry is still present, but we need to find another combination of the generators and change the trigonometric functions sine and cosine to the hyperbolic functions sinh and cosh. As a consequence, the Hamiltonian is no longer a compact generator and thus the quantum system possesses a continuous spectrum only.

The same method can be used to discuss the Noether-symmetry problem of three-dimensional isotropic systems. This time we treat the time translation and space rotation as the trivial symmetry. Similarly we found that there are four kinds of potentials which possess the non-trivial Noether-symmetry: free particle, centrifugal potential (a_{-2}/r^2), harmonic oscillator $(\frac{1}{2}\omega^2 r^2)$ and mixed potential $(a_{-2}/r^2+\frac{1}{2}\omega^2 r^2)$. The minimum non-trivial symmetry is SO(2, 1) × SO(3) instead of SO(2, 1) in the one-dimensional case.

The symmetry larger than the superficial geometric symmetry is called the dynamical symmetry of a quantum system. This symmetry determines the most important properties such as the spacing and degeneracy of the energy level. Since their importance is more obvious in the high-dimensional system, we shall not discuss it in detail here.

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